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A new recipe for the spin characters of the symmetric group

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Abstract

The ring of symmetric functions is a graded ring with important applications in mathematical physics. By examining the various transition matrices between the different bases of the ring of symmetric functions, we are able to write the spin characters of the symmetric group in terms of the ordinary characters of the symmetric group. This approach allows us to describe a new, non-recursive, combinatorial algorithm for the spin characters. We also present simpler algorithms in two special cases.

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1. Introduction

In 1911, Schur [10] described various symmetric functions and introduced the spin characters of the symmetric group in his definitive paper on the projective representation of the symmetric group. The spin characters were not paid much regard until the 1960s when Morris wrote a comprehensive account [6] and gave some recursive formulae for them. More recently, spin characters have been of interest. For example in 1995, Morris [8] gave further improvements and results subsequent to his previous work.

2. Partitions and Young tableaux

A *partition* λ is a finite sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ arranged in weakly descending (meaning non-increasing) order so that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 0$. The components λ_i of the partition λ are called *parts* and the number of parts in a partition λ is called the *length* of the partition, and is denoted by $l(\lambda)$. The sum of the parts $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$ is called the *weight* of the partition and is denoted by $|\lambda|$. We write $\lambda \vdash n$ to mean λ is a partition of weight n. We reserve $\mathcal{P}_n$ to mean the set of all partitions of weight n. The number of occurrences of a part λ_i in a partition λ is called the *multiplicity of* λ_i *in* λ and is denoted by m_{λ_i}. We usually write the multiplicity of each part as a superscript with

$$\lambda = \left(\lambda_1^{m_{\lambda_1}} \lambda_2^{m_{\lambda_2}} \cdots \lambda_j^{m_{\lambda_j}}\right).$$

The *reverse lexicographical ordering* $\mathcal{L}_n$ on the set $\mathcal{P}_n$ of all partitions of $n \in \mathbb{N}$ is the subset of $\mathcal{P}_n \times \mathcal{P}_n$ consisting of all ordered pairs (μ, λ) such that either $\mu = \lambda$ or else the first non-vanishing difference $\mu_i - \lambda_i$ is positive. $\mathcal{L}_n$ is a total ordering and if $(\mu, \lambda) \in \mathcal{L}_n$ we write $\mu \geqslant \lambda$.

Use $\mathcal{S}_n$ to denote the symmetric group whose elements are permutations of n objects. Elements in the same conjugacy class have the same cycle structure, and so the set of partitions of weight n, $\mathcal{P}_n$, partitions the symmetric group into distinct conjugacy classes. If $\pi \in \mathcal{S}_n$ is a permutation with conjugacy class described by $\lambda \in \mathcal{P}_n$ then we use H_λ for the conjugacy class H_π. The number of elements in the class H_λ is denoted by h_λ. Likewise, Z_λ is used for the centralizer Z_π of π and z_λ denotes the number of elements in the centralizer Z_λ.

Lemma 2.1 ([9], proposition 1.1.1)**.** *For a partition $\lambda \vdash n$ of weight n, the number of elements in the centralizer is determined by $z_\lambda = \prod_i \lambda_i^{m_{\lambda_i}} m_{\lambda_i}!$ where the product is taken over all i for which λ_i is a non-zero part of λ with multiplicity m_{λ_i}.*

Lagrange's theorem ([3], theorem 2 4.4) tells us that we can determine the size of each conjugacy class H_λ by the size of the centralizer. Indeed, $|H_\lambda| = |\mathcal{S}_n| / |Z_\lambda|$ and so

$$h_\lambda = \frac{n!}{z_\lambda} = \frac{n!}{\prod_i \lambda_i^{m_{\lambda_i}} m_{\lambda_i}!}. \tag{1}$$

Every partition λ of weight n can be associated with a *Young diagram* Y^λ involving n boxes (cells, circles, dots, etc) with the ith row containing λ_i boxes. The *staircase* of a Young diagram consists of all the boxes in a continuous outer ribbon going from the upper right to the lower left (or vice versa).

A *Young tableau* τ for a partition λ of weight n is an assignment of n numbers (not necessarily all different) to the n boxes of the Young diagram Y^λ. *Standard numbering* means that the assignment of the numbers $1, 2, \ldots, d \leqslant n$ is such that the numbers are strictly increasing from left to right across each row, and strictly decreasing down each column. There are several methods of *semi-standard numbering*. One of them is *unitary numbering* in which the assignment of the numbers $1, 2, \ldots, d \leqslant n$ is such that the numbers are weakly increasing (meaning non-decreasing) from left to right across each row and strictly decreasing down each column. When the numbering is unitary we call the tableau *unitary*.

Another type of semi-standard numbering is *regular numbering* in which the assignment of the numbers $1, 2, \ldots, d \leqslant n$ is such that the numbers are weakly increasing from left to right across each row, weakly increasing down each column and like digits form a continuous staircase of some subdiagram. When the numbering is regular we call the tableau *regular*.

Let λ be a partition with Young diagram Y^λ. We can *inject a partition* $\rho \vdash |\lambda|$ the same weight as λ into the Young diagram Y^λ of λ to form a tableau. To inject a partition $\rho = (\rho_1, \rho_2, \ldots, \rho_d)$ of length d into the diagram, we mean injecting ρ_1 1's; ρ_2 2's; ...; ρ_d d's. A *negative application* is an injection of numbers giving a regular tableau in which like digits occupy an even number of rows.

Associated with each Young tableau is a *word* formed by reading the numbers in the tableau in successive rows from right to left, starting from the top row. The numbers which make up the word are called the *elements* of the word. A *standard word* is one in which the numbers $1, 2, \ldots, n$ each occurs only once. The *indices* of each element of the word are defined recursively by the following steps:

(i) the number 1 has index 0;
(ii) if the number r has index i, the number $r + 1$ has
  (a) index i if it is to the right of r or
  (b) index $i + 1$ if it is to the left of r.

For each standard word w the *charge of the word* $c(w)$ is the sum of the indices of each element of w. For each nonstandard word of a tableau τ, we extract a unique set of standard *subwords* of τ. The extraction is defined recursively by the following steps:

(i) check if w is standard, if not—proceed, otherwise we are done;
(ii) starting from the left, mark the first 1 that occurs in w;
(iii) if r has been marked, then search for the first occurrence of $r + 1$ moving right to the end of the word, and then through the word once more:
  (a) if no $r+1$ is found, delete the marked standard word from w and repeat the procedure from the start to extract another standard word or
  (b) if $r + 1$ is found, mark it and begin the search for $r + 2$.

The *charge of a nonstandard word* is the sum of the charges of its standard subwords.

3. The ring of symmetric functions

A polynomial from the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ of polynomials in n indeterminates is a *symmetric polynomial* if it is invariant under the action of the symmetric group. The set of symmetric polynomials in n indeterminates $\Lambda_n = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}$ forms a subring of the ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ of polynomials in n indeterminates. For $k \geqslant 0$, let $\Lambda_n^{(k)}$ consist of the homogeneous symmetric polynomials of degree k, and include the zero polynomial in each $\Lambda_n^{(k)}$ for all $k \geqslant 0$. Including the zero means that each $\Lambda_n^{(k)}$ is a group under addition. Also, since $\Lambda_n^{(k)}\Lambda_n^{(j)} \subseteq \Lambda_n^{(k+j)}$, the ring of symmetric polynomials

$$\Lambda_n = \bigoplus_{k\geqslant 0} \Lambda_n^{(k)}$$

is graded. Because the ring is graded, we can set up an inverse system of natural projections and take a projective limit of the homogeneous subgroups. To this end we use the natural projection $\rho_{n+1,n} : \Lambda_{n+1} \to \Lambda_n$ making any x_{n+1} terms in Λ_{n+1} equal to zero. Clearly $\rho_{n+1,n}$ is a surjective ring homomorphism. Next, we restrict $\rho_{n+1,n}$ to act on polynomials of degree $k \leqslant n$ by putting $\rho_{n+1,n}^k : \Lambda_{n+1}^{(k)} \to \Lambda_n^{(k)}$ so that $\rho_{n+1,n}^k$ is also injective. This means that $\rho_{n+1,n}^k\big(\Lambda_{n+1}^{(k)}\big) = \Lambda_n^{(k)}$. Taking the projective limit of this inverse system $\Lambda^{(k)} = \lim_{\leftarrow n} \Lambda_n^{(k)}$ gives $\Lambda^{(k)}$, the set of *homogeneous symmetric functions of degree k*, with zero. For comprehensive details on projective limits, see [2].

The $\Lambda^{(k)}$ form additive groups, and we use these groups to construct the graded *ring of symmetric functions* by putting

$$\Lambda = \bigoplus_{k\geqslant 0} \Lambda^{(k)}.$$

We use the natural projection $\rho_n^k : \Lambda^{(k)} \to \Lambda_n^{(k)}$ mapping symmetric functions of degree k to symmetric polynomials of degree k in n indeterminates to describe certain classical symmetric functions.

For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ with $j \leqslant n$ parts, use λ^0 to mean the partition λ with $n - j$ zeros adjoined, so that $\lambda^0 = (\lambda_1, \lambda_2, \ldots, \lambda_j, 0_{j+1}, \ldots, 0_n)$ is a partition of length n. Then the *monomial symmetric polynomial* $m_\lambda(x_1, x_2, \ldots, x_n)$ *in* n *indeterminates* corresponding to the partition λ is the sum over all the distinct monomials in n indeterminates $\{x_1, x_2, \ldots, x_n\}$ with the parts of λ^0 as exponents. The idea is that whenever $j < n$, in each monomial $j - n$ of the indeterminates have the form $x_i^{0_i}$, and vanish, with j indeterminates surviving (with a nonzero exponent). That is, $m_\lambda(x_1, x_2, \ldots, x_n) = \sum x_1^{\lambda_1^0} x_2^{\lambda_2^0} \ldots x_n^{\lambda_n^0}$ where the sum is over all distinct permutations of λ^0, treating all the 0_i parts equally as 0, and putting any $x_i^{0_i} = 1$.

If $\lambda \vdash k$, then $m_\lambda(x_1, x_2, \ldots, x_n)$ is homogeneous of degree k and the *monomial symmetric function* $m_\lambda(x)$ is the unique symmetric function from the group $\Lambda^{(k)}$ of homogeneous symmetric functions which satisfies the projection to n indeterminates $\rho_n^k(m_\lambda(x)) = m_\lambda(x_1, \ldots, x_n)$ for every $n \geqslant k$. The space spanned by all monomial symmetric functions of degree k is $\Lambda^{(k)}$ ([9], proposition 4.3.3), whence $\Lambda = \mathbb{Z}[m_\lambda]$.

There are several other bases for $\Lambda^{(k)}$. We are specifically interested in two of them: the power-sum symmetric functions and the Schur S-functions. For any $r \in \mathbb{N}$, the rth *power-sum symmetric function* is $p_r(x) = \sum_{i \geqslant 1} x_i^r$. The power-sum symmetric functions are multiplicative, so for any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ we write $p_\lambda = p_{\lambda_1} \ldots p_{\lambda_k}$. The power-sum symmetric functions p_λ are well known as a $\mathbb{Q}$-basis for the ring of symmetric functions (see [5], page 16, for example).

A characteristic mapping is an isomorphism from the ring generated by the characters of the symmetric group onto the ring of symmetric functions. Frobenius' theorem says that there is a characteristic mapping which maps the group characters χ^λ of the symmetric group to the symmetric function $s_\lambda(x)$.

Explicitly we have

$$s_\lambda(x) = \sum_{\rho \vdash |\lambda|} z_\rho^{-1} \chi_\rho^\lambda p_\rho(x),$$

where χ_ρ^λ is the character on the class ρ, which has centralizer of size z_ρ; and $p_\rho(x)$ is the power-sum symmetric function. The power-sum symmetric function occurs here definitively. Indeed, the right-hand side of this expression is just the characteristic mapping of χ^λ, the left-hand side being its image in the ring of symmetric functions. The symmetric functions $s_\lambda(x)$ are called *Schur S-functions*. The Schur S-functions form a $\mathbb{Z}$-basis for the ring of symmetric functions ([5], I.3.3) and they establish a strong connection between the theory of symmetric functions and the combinatorial theory of Young diagrams. Indeed, Schur S-functions may be defined purely combinatorially, as in [9].

The transition matrix from the power-sum symmetric functions to the Schur S-functions is just the character table of the symmetric group $\mathcal{S}_n$. This is because of the orthogonality of the characters. This means that

$$p_\rho(x) = \sum_{\lambda \vdash |\rho|} \chi_\rho^\lambda s_\lambda(x), \tag{2}$$

where χ_ρ^λ is the character χ^λ on the class ρ.

The transition matrix K from the Schur S-functions $s_\lambda(x)$ to the monomial symmetric functions $m_\mu(x)$ has coefficients $K_{\lambda\mu}$ in the equation

$$s_\lambda(x) = \sum_{\mu \vdash |\lambda|} K_{\lambda\mu} m_\mu(x).$$

The numbers $K_{\lambda\mu}$ are called *Kostka numbers* and the matrix K is called the *Kostka matrix*. Actually it turns out that there is a broader definition of Kostka numbers and the Kostka matrix, which we meet in section 6. When we do meet the Kostka numbers and Kostka matrix again, we see that there is a t dependence that has not come into play yet. It turns out that the case described here is for when $t = 1$.

4. Projective representations

A *projective representation* of the symmetric group $\mathcal{S}_n$ is a mapping M from $\mathcal{S}_n$ into the general linear group $GL(k)$ so that for $x, y \in \mathcal{S}_n$

$$M(x)M(y) = \tau(x, y)M(xy),$$

for some $\tau(x, y) \in \mathbb{C}^* = \mathbb{C}/\{0\}$. Because the linear transformations are invertible, the mapping τ is a 2-cocycle. The set of cohomology classes of 2-cocycles forms an Abelian group called the *Schur multiplier*. We refer to section 1 of [12] for further details or to [1] for the comprehensive account. In this case, since we are mapping from the symmetric group, the nature of the equivalence classes of these 2-cocycles is available. Indeed, they have a 2-element classification for $n \geqslant 4$ as $\mathbb{Z}_2$ ([1], theorem 2.7). This equivalency is determined by the Schur multiplier. Representations M of $\mathcal{S}_n$ for which $\tau(x, y) \equiv 1$ correspond with the ordinary linear representations; otherwise there exist group elements such that $\tau(x, y) \equiv -1$ and these correspond to a double cover $\tilde{\mathcal{S}}_n$ of $\mathcal{S}_n$. It is the characters of this double cover that we mean when we talk about the irreducible characters of the projective representation. This double cover is sometimes called the *spin representation* and its characters are called *spin characters*. We use this terminology, and in section 6 we write the spin characters in terms of the ordinary ones. We use this relationship to describe a new combinatorial algorithm to determine spin character values.

Some specifics on spin characters, in the context of Q-functions, are noted here. Denote by ζ^λ_μ the spin character ζ^λ on the class μ of the symmetric group $\mathcal{S}_n$. Use $\mathcal{OP}$ to mean the class of partitions with all parts odd integers, and call the members of $\mathcal{OP}$ *odd part partitions*. Use $\mathcal{DP}$ to mean the class of partitions with all parts distinct integers (so that the parts are written in strict descending order) and call the members of $\mathcal{DP}$ *distinct part partitions*. Only spin characters ζ^λ with $\lambda \in \mathcal{DP}$ on the class $\rho \in \mathcal{OP}$ are relevant here, consistent with the same restrictions in [10] and [1].

Just as Frobenius had shown that the ordinary group characters of the symmetric group mapped to S-functions, Schur called the characteristic mapping of the spin characters Q-functions.

For any $\lambda \in \mathcal{DP}$, the *Schur Q-function* Q_λ is determined by the characteristic mapping of spin characters ζ^λ so that

$$Q_\lambda(x) = \sum_{\substack{\rho \vdash |\lambda| \\ \rho \in \mathcal{OP}}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\rho^{-1} \zeta^\lambda_\rho p_\rho(x), \tag{3}$$

where $l(\lambda)$ means the length of the partition λ; ζ^λ_ρ are the spin characters on the class $\rho \in \mathcal{OP}$; $p_\rho(x)$ are the power-sum symmetric functions; and, z_ρ is the size of the centralizer, determined by Frobenius' formula (see lemma 2.1).

5. Hall–Littlewood functions

The Hall–Littlewood functions are defined in the ring $\Lambda[t]$ of symmetric functions with coefficients in $\mathbb{Z}[t]$. The *Hall–Littlewood P-polynomials* are given by

$$P_\lambda(x_1\cdots x_n;t)=\sum_{\omega\in\mathcal{S}_n/\mathcal{S}_n^\lambda}\omega\left(x_1^{\lambda_1}\cdots x_n^{\lambda_n}\prod_{\lambda_i>\lambda_j}\frac{x_i-tx_j}{x_i-x_j}\right),$$

where $\mathcal{S}_n^\lambda$ is the subgroup of permutations $\omega\in\mathcal{S}_n$ such that $\lambda_{\omega(i)}=\lambda_i$. We pass to the inverse limit requiring that, for every $\lambda\vdash k$, the image of the *Hall–Littlewood functions* $P_\lambda(x;t)$ from the subgroup $\Lambda^{(k)}[t]$ be the Hall–Littlewood polynomials $P_\lambda(x_1,\ldots,x_n;t)$ in $\Lambda_n^{(k)}[t]$ for each $n\geqslant k$. The Hall–Littlewood (HL) P-functions are algebraically independent over $\mathbb{Z}[t]$ and form a $\mathbb{Z}[t]$-basis for the ring $\Lambda[t]$ ([5], proposition 3.2.7). When $t=1$ the HL P-functions are just the monomial symmetric functions: $P_\lambda(x;1)=m_\lambda(x)$.

The *Hall–Littlewood Q-functions* are scalar multiples of HL P-functions given by

$$Q_\lambda(x;t)=b_\lambda(t)P_\lambda(x;t), \tag{4}$$

where $b_\lambda(t)=\prod_{i\geqslant1}\varphi_{m_{\lambda_i}}(t)$ and

$$\varphi_r(t)=(1-t)(1-t^2)\cdots(1-t^r), \tag{5}$$

with m_{λ_i} being the multiplicity of the part λ_i in λ.

When $t=-1$, equation (5) vanishes for any $r\geqslant2$. This means that the multiplicity m_{λ_i} of any part λ_i must equal 1. In other words, the HL Q-function $Q_\lambda(x;-1)$ is nonzero only for distinct part partitions. Indeed, the HL Q-functions $Q_\lambda(x;-1)$ are exactly the Q-functions introduced by Schur and the subject of section 4. This provides an important link to the spin characters because the Q-functions are given definitively by a characteristic mapping (equation (3)). Note also that when $t=-1$

$$b_\lambda(-1)=\prod_{i\geqslant1}\varphi_{m_{\lambda_i}}(-1)=\prod_{i\geqslant1}(1+1)=2^{l(\lambda)}, \tag{6}$$

since necessarily $\lambda\in\mathcal{DP}$, and equation (4) becomes

$$Q_\lambda(x;-1)=2^{l(\lambda)}P_\lambda(x;-1).$$

The transition matrix $X(t)$ between the power-sum symmetric functions and the HL P-functions has coefficients $X_\rho^\lambda(t)$ determined by

$$p_\rho(x)=\sum_\lambda X_\rho^\lambda(t)P_\lambda(x;t). \tag{7}$$

When $t=0$ the HL P-functions are the Schur S-functions: $P_\lambda(x;0)=s_\lambda(x)$, and so the entries in the transition matrix $X(0)$ are the ordinary group characters of the symmetric group (see equation (2))

$$X_\rho^\lambda(0)=\chi_\rho^\lambda.$$

Lemma 5.1. *When $t=-1$ and ρ is an odd part partition, the entries in the transition matrix $X(t)$ between the power-sum symmetric functions and the HL P-functions are scalar multiples of the spin characters of the symmetric group. Specifically, for $\rho\in\mathcal{OP}$*

$$X_\rho^\lambda(-1)=2^{\frac{1}{2}[l(\lambda)-l(\rho)+\epsilon]}\zeta_\rho^\lambda \tag{8}$$

$$\zeta_\rho^\lambda=2^{\frac{1}{2}[l(\rho)-l(\lambda)-\epsilon]}X_\rho^\lambda(-1). \tag{9}$$

Proof. The orthogonality relations developed in section 3.7 of MacDonald's book [5] yield

$$Q_\lambda(x;t)=\sum_\rho z_\rho(t)^{-1}X_\rho^\lambda(t)p_\rho(x), \tag{10}$$

where $z_\rho(t)$ is a generalized form of Frobenius' formula for the size of the centralizer, given by

$$z_\rho(t) = z_\rho \prod_{i\geqslant 1} (1 - t^{\rho_i})^{-1}. \tag{11}$$

When ρ is an odd part partition and $t = -1$, equation (11) yields

$$z_\rho(-1) = z_\rho 2^{-l(\rho)}.$$

Using this, we evaluate equation (10) at $t = -1$ to obtain

$$Q_\lambda(x; -1) = \sum_\rho z_p^{-1} 2^{l(\rho)} X_\rho^\lambda(-1) p_\rho(x). \tag{12}$$

Since $Q_\lambda(x; -1)$ is Schur's Q-function, we can compare this equation to Schur's original equation introducing the Q-functions (equation (3)):

$$Q_\lambda(x) = \sum_{\rho\in\mathcal{OP}} 2^{\frac{1}{2}(l(\lambda)+l(\rho)+\epsilon)} z_\rho^{-1} \zeta_\rho^\lambda p_\rho(x).$$

Comparing coefficients gives equation (8). Re-arranging to make the spin character the object gives equation (9). □

6. A new recipe for the spin characters of the symmetric group

A rich and well-established connection between the theory of symmetric functions and the combinatorial properties of Young diagrams and tableaux enables us to write a new combinatorial algorithm for calculating the spin characters. The algorithm we describe is just an amalgamation of two existing theorems/algorithms: the first is due to Lascoux and Schützenberger and appears shortly (theorem 6.1); the second is Schensted's build up staircase algorithm for the ordinary characters of the symmetric group. Appendix A gives the details of Schensted's algorithm.

The *Kostka matrix* $K(t)$ is the transition matrix between the Schur S-functions and the HL P-functions and has coefficients $K_{\lambda\mu}(t)$, called *Kostka numbers*, in the equation

$$s_\lambda(x) = \sum_\mu K_{\lambda\mu}(t) P_\mu(x; t).$$

When $t = 1$, recall that the HL P-functions are just the monomial symmetric functions. In this case, the Kostka numbers and Kostka matrix just described are the same as those given in section 3. We are interested in the case $t = -1$. Lascoux and Schützenberger found a combinatorial formula for the Kostka numbers for any value of t.

Theorem 6.1 (Theorem of Lascoux and Schützenberger, [5], theorem 3.6.5)**.** *The elements of the Kostka Matrix are given by*

$$K_{\lambda\rho}(t) = \sum_\tau t^{c(\tau)},$$

where the sum is over all possible unitary tableaux τ formed by injecting ρ into the Young diagram Y^λ, and $c(\tau)$ is the charge of the word associated with the tableau τ.

The Kostka matrix $K(t)$ can be used to connect the transition matrix $X(t)$ and the ordinary characters. Specifically ([5], equation 3.7.6')

$$X_\rho^\lambda(t) = \sum_{\mu\geqslant\lambda} \chi_\rho^\mu K_{\mu\lambda}(t), \tag{13}$$

where the sum is over all partitions $\mu \geqslant \lambda$ in the reverse lexicographical ordering.

Theorem 6.2. *Suppose $\lambda \in \mathcal{DP}$ is a distinct part partition of weight n with length $l(\lambda)$, and $\rho \in \mathcal{OP}$ is an odd part partition of weight n and length $l(\rho)$. The spin character ζ^λ on the class ρ is*

$$\zeta^\lambda_\rho = 2^{\frac{1}{2}[l(\rho)-l(\lambda)-\epsilon]} \sum_{\mu \geqslant \lambda} \chi^\mu_\rho K_{\mu\lambda}(-1), \tag{14}$$

where the sum is over all partitions μ greater than or equal to λ in the reverse lexicographical ordering; χ^μ is the ordinary character of the symmetric group $\mathcal{S}_n$ on the class ρ; $K_{\mu\lambda}(-1)$ are the Kostka numbers with $t = -1$ and ϵ is appropriately 0 or 1.

Proof. Using equation (9) from lemma 5.1 with equation (13) is all that is required. □

Remark 6.3. Theorem 6.2 allows us to determine the spin characters using known combinatorial methods for calculating the ordinary characters. We give a 'build-up' method here because it fits well with Lascoux and Schützenberger's algorithm for the Kostka numbers. This means that for larger order characters we do not rely on needing to know the spin characters of lower orders.

Algorithm 6.4. *Suppose $\lambda \in \mathcal{DP}$ is a distinct part partition of weight n and $\rho \in \mathcal{OP}$ is an odd part partition of weight n. To calculate the spin character ζ^λ_ρ of the symmetric group $\mathcal{S}_n$, we must consider partitions $\mu \vdash |\lambda|$ where $\mu \geqslant \lambda$ in the reverse lexicographic ordering. For each of these μ,*

(i) calculate the charge $c(\tau)$ of each of the unitary tableaux τ formed by injecting λ into Y^μ. Compute the sum $\sum_\tau (-1)^{c(\tau)}$;

(ii) calculate the number of negative applications of each of the regular tableaux σ formed by injecting ρ into Y^μ. Denote by $n_e(\sigma_\mu)$ the number of tableaux σ which involves an even number of negative applications and by $n_o(\sigma_\mu)$ the number of tableaux σ which involves an odd number of negative applications. Find the difference $n_e(\sigma_\mu) - n_o(\sigma_\mu)$, and call this difference $\Delta n(\sigma_\mu)$.

Then

$$\zeta^\lambda_\rho = 2^{\frac{1}{2}[l(\rho)-l(\lambda)-\epsilon]} \sum_{\mu \geqslant \lambda} \left[\Delta n\left(\sigma_\mu\right) \cdot \left(\sum_\tau (-1)^{c(\tau)} \right) \right], \tag{15}$$

where τ has shape μ; $l(\rho)$ and $l(\lambda)$ denote the lengths of the partitions ρ and λ respectively; and ϵ is 1 if $l(\rho) - l(\lambda)$ is odd, and 0 otherwise.

Proof. Part (i) of our algorithm is exactly Lascoux and Schützenberger's algorithm for the Kostka numbers $K_{\mu\lambda}(-1)$. Part (ii) of our algorithm is exactly Schensted's build-up staircase recipe for the ordinary characters which is detailed in appendix A. Lascoux and Schützenberger theorem (theorem 6.1) requires tableaux of shape μ. Since Schensted's build-up staircase recipe also requires tableaux of shape μ, we naturally merge the two algorithms. Indeed, equation (15) is just equation (14) with $K_{\mu\lambda}(-1)$ replaced by $\sum_\tau (-1)^{c(\tau)}$ using Lascoux and Schützenberger's theorem (theorem 6.1) and with χ^μ_ρ replaced by $\Delta n(\sigma_\mu)$ using Schensted's recipe (algorithm A.1). □

Example 6.5. Suppose we want to calculate the spin character $\zeta^{(42)}$ on the class $\rho = (31^3)$. Then we must consider all partitions μ of weight 6 such that $\mu \geqslant \lambda = (42)$. So we put

<table>
<tr><th></th><th>Unitary Tableaux
Inject $\lambda = (42)$</th><th>Regular Tableaux
Inject $\rho = (31^3)$</th><th></th></tr>
<tr><td>$\mu = (6)$</td><td>1 1 1 1 2 2
$w = \{221111\}$
$w = \{2_1 1_0; 2_1 1_0; 1_0; 1_0\}$
$c(w) = 2$
$(-1)^2 = 1$</td><td>1 1 1 2 3 4
$\#$ neg $= 0$
$n_e(\sigma_\mu) = 1$ $n_o(\sigma_\mu) = 0$
$\Delta n(\sigma_\mu) = 1 - 0 = 1$</td><td>$1 \times 1 = 1$</td></tr>
<tr><td>$\mu = (51)$</td><td>1 1 1 1 2
2
$w = \{211112\}$
$w = \{1_0 2_0; 2_1 1_0; 1_0; 1_0\}$
$c(w) = 1$
$(-1)^1 = -1$</td><td>1 1 1 2 3 / 4 $\#$ neg $= 0$
1 1 1 2 4 / 3 $\#$ neg $= 0$
1 1 1 3 4 / 2 $\#$ neg $= 0$
1 1 2 3 4 / 1 $\#$ neg $= 1$
$n_e(\sigma_\mu) = 3$ $n_o(\sigma_\mu) = 1$
$\Delta n(\sigma_\mu) = 3 - 1 = 2$</td><td>$-1 \times 2 = -2$</td></tr>
<tr><td>$\mu = (42)$</td><td>1 1 1 1
2 2
$w = \{111122\}$
$w = \{1_0 2_0; 1_0 2_0; 1_0; 1_0\}$
$c(w) = 0$
$(-1)^0 = 1$</td><td>1 1 1 2 / 3 4 $\#$ neg $= 0$
1 1 2 3 / 1 4 $\#$ neg $= 1$
1 1 1 3 / 2 4 $\#$ neg $= 0$
1 1 3 4 / 1 2 $\#$ neg $= 1$
1 1 1 4 / 2 3 $\#$ neg $= 0$
1 1 2 4 / 1 3 $\#$ neg $= 1$
$n_e(\sigma_\mu) = 3$ $n_o(\sigma_\mu) = 3$
$\Delta n(\sigma_\mu) = 3 - 3 = 0$</td><td>$1 \times 0 = 0$</td></tr>
<tr><td></td><td colspan="2">Also $l(\rho) = 4$ and $l(\lambda) = 2$. Hence
$\zeta^{(42)}_{(31^3)} = 2^{\frac{4-2}{2}} \times -1 = -2.$</td><td>$1 + -2 + 0$
$= -1$</td></tr>
</table>

Figure 1. Diagrams and calculations for example 6.5.

$\mu_1 = (6)$; $\mu_2 = (51)$; $\mu_3 = (42)$ and sum over these μ_i. Figure 1 contains a table in which each row is indexed by each one of the μ_i just listed. The left-hand column in the table shows the sum $\sum_\tau (-1)^{c(\tau)}$ from part (i) of algorithm 6.4. The middle column shows the calculation of the differences $\Delta(\sigma_\mu)$ described in part (ii) of algorithm 6.4. In the right-most column multiples of the two are summed, producing a total of -1 to be multiplied by $2^{\frac{1}{2}(l(\rho)-l(\lambda)-\epsilon)}$. In this case we calculate $\zeta^{(42)}_{(31^3)} = -2$.

Spin character tables are provided in appendix B. Referring to the table for characters of degree 6, we look for the row indexed by $\langle 42\rangle$ and the column headed (31^3). The entry there is the value of the spin character $\zeta^{(42)}_{(31^3)}$ and this value is -2.

7. Special cases

7.1. Length one partitions

Theorem 7.1. *Suppose $\lambda = (n)$ is a length one partition of weight n. The spin character $\zeta_\rho^{\lambda=(n)}$ on the class ρ is*

$$\zeta_\rho^{\lambda=(n)} = 2^{\frac{1}{2}[l(\rho)-\delta]}, \tag{16}$$

where

$$\delta = \begin{cases} 1 & \text{whenever } n \text{ is odd} \\ 2 & \text{whenever } n \text{ is even.} \end{cases}$$

Proof. Consider the special case that $\lambda = (n)$ is a length one partition of weight n. Using our recipe (algorithm 6.4) we need to consider only $\mu \geqslant \lambda$. Since $\lambda = (n)$, the only μ satisfying this requirement is $\mu = (n)$. This means that the sum over μ in equation (15) reduces to the product of $\sum_\tau (-1)^{c(\tau)}$ determined in step (i) and $\Delta n(\sigma_\mu)$ in step (ii) of algorithm 6.4, both evaluated at $\mu = (n)$.

Whenever $\mu = (n)$ and $\lambda = (n)$, the one and only unitary tableau formed by the injection of λ into Y^μ is the trivial one row tableau with 1s everywhere. So the extracted word $w = (111\ldots1)$ of length n is a standard word with charge 0. Hence we always have $\sum_\tau (-1)^{c(\tau)} = (-1)^0 = 1$. This reduces the spin character calculation to the product of $\Delta n(\sigma_\mu)$ and a power of 2.

Next, $\Delta n(\sigma_\mu)$ in step (ii) of algorithm 6.4 depends on the number of negative applications involved in injecting the class ρ into Y^μ. This is a trivial calculation in this case since the number of rows in Y^μ is 1, meaning the number of negative applications must always be 0, since we will never have an even number of rows. There is always one and only one way to inject any partition ρ into a one-row diagram to give a regular tableau. Indeed, we always have only one trivial even negative application in this case. And so we always obtain 1 even and 0 odd negative applications for any ρ, whence $\Delta n(\sigma_\mu) = n_e(\sigma_\mu) - n_o(\sigma_\mu) = 1$.

Hence, in the special case that $\lambda = (n)$, equation (15) always gives $\Delta n(\sigma_\mu) \cdot \left(\sum_\tau (-1)^{c(\tau)}\right) = 1$ summed only once in the case $\mu = (n)$, and so the spin character is just $2^{\frac{1}{2}(l(\rho)-l(\lambda)-\epsilon)}$. Since $l(\lambda) = 1$, this expression reduces further to

$$\zeta_\rho^n = 2^{\frac{1}{2}(l(\rho)-1-\epsilon)}.$$

Next, combine the constants 1 and ϵ (where $\epsilon = 0$ or 1 appropriately) by putting $-1-\epsilon = -\delta$ where the value of δ is

$$\delta = \begin{cases} 1 & \text{whenever } \mathrm{l}(\rho) \text{ is odd} \\ 2 & \text{whenever } \mathrm{l}(\rho) \text{ is even.} \end{cases}$$

Finally, note that since $\rho \vdash n$ is an odd part partition, if n is even, then ρ must have an odd number of parts. Likewise, when n is odd, ρ must have an even number of parts. This means that we can describe δ in terms of n and remove the dependence on $l(\rho)$. □

Example 7.2. The first row of the spin character tables (see appendix B) correspond to the irreducible character $\zeta^{\lambda=(n)}$. Using corollary 7.1, we can easily give the first row of any spin character table for different n. Consider the case when, say, $n = 10$. Since n is even, the value of δ here is $\delta = 2$.

$\zeta_\rho^{(10)} = 2^{\frac{1}{2}(l(\rho)-2)}$

class :	(91)	(73)	(71^3)	(5^2)	(531^2)	(51^5)	(3^31)	(31^7)	(1^{10})
$l(\rho)$	2	2	4	2	4	6	4	8	10
$\zeta_\rho^{(10)}$	1	1	2	1	2	4	2	8	16

Looking at the spin character tables given in appendix B, we see that all the characters just calculated are indeed true and correct.

Remark 7.3. A similar form of equation (16) from theorem 7.1 is mentioned by Morris [6] as having appeared in Schur's original paper [10]. We have, of course, obtained this result in a completely different way to that of Schur.

7.2. When the class is the identity element

Distinct part partitions $\lambda = (\lambda_1, \ldots, \lambda_k)$ of length k can be represented by another type of diagram $\tilde{\mathcal{Y}}^\lambda$ with a main diagonal of k boxes marked with a $\star$ say. Put λ_i boxes in the ith row to the right of the marked box for each $1 \leqslant i \leqslant k$. Then put $\lambda_i - 1$ dashed boxes in the ith column below the marked box for each $1 < i \leqslant k$. The diagram $\tilde{\mathcal{Y}}^\lambda$ constructed in this way is called the *shifted symmetric diagram*. The boxes to the right of the marked diagonal make up the standard *shifted diagram*, denoted here by $\mathcal{Y}^\lambda$.

Graphically, the *hook length* of a partition λ at coordinate x in the Young diagram Y^λ is the number of boxes along the row to the right of x, plus the number of boxes down the column below x, plus 1 (for the box x itself). For $\lambda \in \mathcal{DP}$ with shifted diagram $\mathcal{Y}^\lambda$, the hook length $h(x)$ at $x \in \mathcal{Y}^\lambda$ is defined to be the hook length at x in the shifted symmetric diagram $\tilde{\mathcal{Y}}^\lambda$. We write $h(x)$ as $\tilde{h}(x)$ to clarify this point.

Proposition 7.4 ([5], p 134)**.** *When the class ρ is restricted to the identity element (1^n) of the symmetric group $\mathcal{S}_n$, the coefficients in the transition matrix $X^\lambda_{(1^n)}$ from the power-sum symmetric functions $p_{\rho=(1^n)}$ to the HL P-functions $P_{\lambda\vdash n}(x; t = -1)$ can be calculated using the hook lengths of the shifted diagram $\mathcal{Y}^\lambda$. Explicitly*

$$X^\lambda_{(1^n)}(-1) = \frac{n!}{\prod_{x\in\mathcal{Y}^\lambda} \tilde{h}(x)},$$

where $\tilde{h}(x)$ is the hook length in $\tilde{\mathcal{Y}}^\lambda$ at x for all x in $\mathcal{Y}^\lambda$.

Using this hook-length formula and making use of equation (9) yields the following corollary.

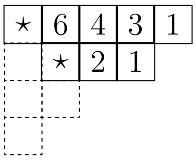
Corollary 7.5. *Let $\rho \vdash n$ and $\lambda \vdash n$ be partitions of weight n with $\rho = (1^n)$. Then the spin character $\zeta^\lambda_{1^n}$ on the class $\rho = (1^n)$ is determined by*

$$\zeta^\lambda_{(1^n)} = 2^{\frac{1}{2}[n-l(\lambda)-\epsilon]} \frac{n!}{\prod_{x\in\mathcal{Y}^\lambda} \tilde{h}(x)},$$

where $\mathcal{Y}^\lambda$ is the shifted diagram of λ; $\tilde{h}(x)$ is the hook length in $\tilde{\mathcal{Y}}^\lambda$ at x for all x in $\mathcal{Y}^\lambda$; and ϵ is 0 or 1 accordingly.

Proof. Combining equation (9) from lemma 5.1 with the hook-length formula given in proposition 7.4 completes the proof. □

Example 7.6. Consider the class (1^6) of $\mathcal{S}_6$. For $\lambda = (42)$ we have $l(\lambda) = 2$. The diagram $\tilde{\mathcal{Y}}^\lambda$ of λ with the hook lengths for each coordinate $x \in \mathcal{Y}^\lambda$ injected at each $x \in \mathcal{Y}^\lambda$ is



and so

$$\zeta_{(1^6)}^{\lambda=(42)} = 2^{\frac{1}{2}(n-l(\lambda)-\epsilon)} \frac{n!}{\prod_{x\in\mathcal{Y}^\lambda} \tilde{h}(x)} = 2^{\frac{1}{2}(6-2-\epsilon)} \frac{6!}{6\times 4!} = 2^{\frac{1}{2}(4)}5 = 20,$$

where the appropriate value for ϵ is $\epsilon = 0$.

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Appendix A. Ordinary characters of the symmetric group

Schenstead [11] gives an excellent description of a recursive staircase algorithm for calculating the ordinary characters of the symmetric group 'based upon a famous formula due to Frobenius' ([11], p 142). From this Schenstead goes on to give a build-up form of this algorithm. We employ Schensted's build-up algorithm in part (ii) of our algorithm 6.4.

Algorithm Appendix A.1 (Schensted's build-up staircase algorithm, [11], 3.5.3)**.** *To calculate the character χ_ρ^λ on the class $\rho = (\rho_1, \rho_2, \ldots, \rho_k)$ in the irreducible representation $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, first draw the Young diagram Y^λ; then inject ρ in a regular manner, in which like digits must form a continuous staircase of some subdiagram. The character χ_μ^λ is equal to the number of ways of doing the above that involve an even number of negative applications minus the number of ways of doing the above that involve an odd number of negative applications.*

Appendix B. Spin character tables

Spin character tables from [1] and [6] up to degree 10. The rows are indexed by the characters of the irreducible representations ζ^λ in distinct part partitions; the columns are headed above by the class ρ, an odd part partition. For example, to look up the character $\zeta^{(41)}$ on the class (31^2), we go to the table 'degree 5', read down the left hand side to the character $\chi^{(41)}$, written as $\langle 41\rangle$, and then across to the column headed by (31^2). The entry 0 found there is the value of the spin character $\zeta_{31^2}^{(41)}$.

Degree 4

class →	(1^4)	(31)
$\langle 4\rangle$	2	1
$\langle 31\rangle$	4	-1

Degree 5

class →	(1^5)	(31^2)	(5)
$\langle 5\rangle$	4	2	1
$\langle 41\rangle$	6	0	-1
$\langle 32\rangle$	4	-1	1

Degree 6

class →	(1^6)	(31^3)	(51)	(3^2)
$\langle 6\rangle$	4	2	1	1
$\langle 51\rangle$	6	2	-1	-2
$\langle 42\rangle$	20	-2	0	2
$\langle 321\rangle$	4	-1	1	-2

Degree 7

class →	(1^7)	(31^4)	(51^2)	(3^21)	(7)
$\langle 7\rangle$	8	4	2	2	1
$\langle 61\rangle$	20	4	0	-1	1
$\langle 52\rangle$	36	0	-1	0	1
$\langle 43\rangle$	20	-2	0	2	-1
$\langle 421\rangle$	28	-4	2	-2	0

Degree 8

class →	(1^8)	(31^5)	(51^3)	(3^21^2)	(71)	(53)
$\langle 8\rangle$	8	4	2	2	1	1
$\langle 71\rangle$	48	12	2	0	-1	-2
$\langle 62\rangle$	112	8	-2	-2	0	2
$\langle 53\rangle$	112	-4	-2	4	0	1
$\langle 521\rangle$	64	-4	1	-2	1	-1
$\langle 431\rangle$	48	-6	2	0	-1	1

Degree 9

class →	(1^9)	(31^6)	(51^4)	(3^21^2)	(71^2)	(531)	(3^3)	(9)
$\langle 9\rangle$	16	8	4	4	2	2	2	1
$\langle 81\rangle$	56	16	4	2	0	-1	-2	-1
$\langle 72\rangle$	160	20	0	-2	-1	0	2	1
$\langle 63\rangle$	224	4	-4	2	0	1	1	-1
$\langle 54\rangle$	112	-4	-2	4	0	-1	-4	1
$\langle 621\rangle$	240	0	0	-6	2	0	-6	0
$\langle 531\rangle$	336	-24	4	0	0	-1	6	0
$\langle 432\rangle$	96	-12	4	0	-2	2	-6	0

Degree 10

class →	(1^{10})	(31^7)	(51^5)	(3^21^4)	(71^3)	(531^2)	(3^31)	(91)	(73)	(5^2)
$\langle 10\rangle$	16	8	4	4	2	2	2	1	1	1
$\langle 91\rangle$	128	40	12	8	2	0	-2	-1	-2	-2
$\langle 82\rangle$	432	72	8	0	-2	-2	0	0	2	2
$\langle 73\rangle$	768	48	-8	0	-2	2	6	0	-1	-2
$\langle 64\rangle$	672	0	-12	12	0	0	-6	0	0	2
$\langle 721\rangle$	400	20	0	-8	1	0	-4	1	-1	0
$\langle 631\rangle$	800	-20	0	-4	2	0	1	-1	1	0
$\langle 541\rangle$	448	-28	2	4	0	-2	2	1	0	-2
$\langle 532\rangle$	432	-36	8	0	-2	1	0	0	-1	2
$\langle 4321\rangle$	96	-12	4	0	-2	2	-6	0	2	-4

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